Early form. More on disc* 1/3

GLIMPSES OF TWENTIETH CENTURY MATHEMATICS.

At the time when "modern mathematics" was a frequently heard phrase, there were two common misconceptions as to what it meant. Just as the motor-car was quite different from the horse and cart, and had made the older form of transport practically obsolete, it was thought that modern mathematics had no connection with traditional mathematics, which indeed it replaced and made unnecessary.

Both these ideas are completely incorrect. Modern mathematics grew out of the older variety, and one of its most valuable uses is to throw light on some problem that can only be expressed in the symbolism of the older subjects.

VECTORS.

Modern mathematics is described as general and abstract. It is abstract; it does not deal with the whole of any situation, but only with some aspect of it. It is general; the aspect in question may be seen in many essentially different situations.

We can illustrate this by considering the modern use of the word "vector". Originally, "vector" had a very definite meaning. A vector could be represented by a line segment, AB, in 2 or 3 dimensions, with an arrow to indicate the direction from A to B. It had applications to displacements, velocities, accelerations and forces.

Vectors had certain properties that made them interesting to study. In modern usage, the word "vector" is applied to anything that has these properties. Thus it does not indicate any particular type of object; it refers to a certain aspect of many situations.

The properties we require are very few. A vector can be multiplied by a number; two vectors can be added. Any collection of mathematical objects in which these two operations can be defined is called "a vector space". Of course the definitions of addition and multiplication cannot be completely random. They must be such that, when working with them, we can forget that they are not our usual addition and multiplication, and even so, we shall not be led to incorrect results. In a formal treatise this requirement would be spelt out in a number of axioms.

OPTIONAL PROPERTIES.

In the original usage of vectors, lengths and angles played a part. It would be possible include in our description of what we meant by "vector space" the requirement that length and angle should be defined. Indeed we shall soon bring in such requirements, but these will then be regarded as singling out a special (though admittedly important) type of vector space. For the present, however, we are not doing this. We are agreeing to accept a situation in which these concepts are lacking as a vector space.

An extreme example would be if we supposed that marking the point (x,y) on graph paper indicated that we were thinking of x cats and y dogs. It would not matter if the graph paper was marked out in squares or in

*"disc" must refer to his computer - none of that work was saved

parallelograms. There is no reason for saying that a cat must be perpendicular to a dog. There might be some objection to calling this example a vector space, since the numbers multiplying vectors are supposed to include fractions and negative numbers, which are not appropriate when applied to animals. However perhaps this rather crude illustration will serve to emphasise that the basic concept of vector space includes systems with no resemblance at all to anything in Euclidean geometry.

THEOREMS ON VECTOR SPACES.

It might seem that there is so little information in the definition that there will be nothing at all to say about the most general vector space, but this is not so. We can, for instance, prove that in such a space the diagonals of a parallelogram bisect each other.

To begin with, we can attach a meaning to two vectors having the same direction. We say that k**u** has the same direction as **u**, and that the line joining the origin to **u** consists of all the vectors k**u**. The line through **v** parallel to the line just mentioned consists of all the vectors **v**+k**u**. The points **v**, **v**+**u**, **v**+2**u**, **v**+3**u**, and so on, are evenly spaced along this line.

In particular, if **w**=**v**+2**u**, then **v**+**u** is the mid-point between **u** and **w**, and is given by (1/2)(**v**+**w**).

The points **0**, **a**, **b**, and **a**+**b** are corners of a parallelogram. The mid-point of **0** and **a**+**b** is (1/2)(**a**+**b**), and this is also the mid-point of **a** and **b**, which proves the theorem mentioned above.

EXAMPLES OF VECTOR SPACES.

We now look at some examples of vector spaces. We begin with two very familiar examples, then we have a less familiar example, and after that something totally unexpected.

(i). Space of 2 dimensions. A vector is specified by two numbers. If **u** = (u_1, u_2) and **v** = (v_1, v_2), then we define k**u** as (ku_1, ku_2) and **u+v** as (u_1+v_1, u_2+v_2).

(ii). Space of 3 dimensions. In the same way we define k**u** as (ku_1, ku_2, ku_3) and **u+v** as $(u_1+v_1, u_2+v_2, u_3+v_3)$.

(iii). Space of n dimensions. For the physical space in which we live and move, the numbers 2 and 3 have special significance. However, in the mathematical patterns used in (i) and (ii), nothing is done that depends on the particular numbers 2 and 3. We can define a vector in space of n dimensions, specified by n numbers. To multiply such a vector by k, we multiply each of the numbers specifying it by k. To get the sum of two such vectors, we add the corresponding numbers in the two brackets.

(iv). Function space. There are simple procedures for multiplying a function by a number and for adding two functions. These have the pattern we require in a vector space.

For example, if **s** stands for the function $s(x) = \sin x$, then it is natural to interpret **t=3s** as meaning that $\mathbf{t}(x) = 3 \sin x$. If **c** stands for the cube function, with $c(x) = x^3$, then we interpret **f=s+c** as meaning $f(x)=\sin(x)+x^3$.

Thus the procedures we use, almost unconsciously, when making calculations with functions are of the type ap~~p~~propriate to work with vectors. We may speak of functions as vectors, and all the functions defined on a given interval as forming a vector space.

Functions that can be expanded in a power series are known as analytic. For such functions there is a very close analogy with the n-dimensional vectors considered in (iii).

For example

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots.$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

$$(1-x)^{-1} + (1-x)^{-2} = 2 + 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5$$

In the top row we see the coefficients (1, 1, 1, 1, 1, 1, ...). In the second row we see (1, 2, 3, 4, 5, 6, ...). If we add these by the usual vector rule we get (2, 3, 4, 5, 6, 7, ...) as seen in the bottom row. Thus we may think of these series as specifying vectors in space of infinite dimensions, and as being added in the way usual for vectors.